

## PRIME IDEALS AND TOPOS POINTS OF MONOIDS

ILIA PIRASHVILI

ABSTRACT. Let  $M$  be a commutative monoid such that  $M/M^*$  is finitely generated, where  $M^*$  is the group of invertible elements of  $M$ . We will show that for such monoids, the points of the topos of  $M$ -sets are in a one-to-one relation with the prime ideals of  $M$ .

## 1. INTRODUCTION

A Topos  $\mathcal{T}$  is a category, equivalent to the category of sheaves of sets over a site  $\mathbf{T}$ . As such, it is of fundamental importance. Some examples of topoi include the category of  $G$ -sets (denoted by  $\mathcal{S}_G$  in this paper) for a given group  $G$ , or more generally  $M$ -sets ( $\mathcal{S}_M$ ), where  $M$  is a monoid.

In topos theory, there exists a general notion of points, called topos points, and they are denoted by  $\mathbf{Pts}(\mathcal{T})$ . As the name would indicate, they are a core construction for any topos. As an example, for any group  $G$ , the category  $\mathbf{Pts}(\mathcal{S}_G)$  is equivalent to the category with a single object  $x$ , with  $\text{Aut}(x) = G$ . This has been known for a long time.

In this paper, we will study the topos points of  $\mathcal{S}_M$ , where  $M$  is an 'almost-finitely generated' commutative monoid. This class contains both arbitrary abelian groups, as well as finitely generated commutative monoids. We will show that in this case, the topos points correspond exactly to the localisations of  $M$  with its prime ideals  $\mathfrak{p} \in \text{Spec}(M)$ . In particular, there is a one-to-one relation between the isomorphism class of the topos points of  $\mathcal{S}_M$  and the actual points of the topological space  $\text{Spec}(M)$ .

In addition to the conceptual nature of this result, it also significantly simplifies the calculation of  $\mathbf{Pts}(\mathcal{S}_M)$ , as it reduces it to calculating  $\text{Spec}(M)$ . This, on the other hand, is very simple if the monoid  $M$  is given in terms of generators and relations, using the results obtained in [4].

## 2. MONOIDS AND TOPOI

It is well known that the categories  ${}_M\mathcal{S}$  and  $\mathcal{S}_M$  of left and right  $M$ -sets form topoi [2]. In particular, this means that they are closed with respect to all limits and colimits. For example, the coproduct in  $\mathcal{S}_M$  is given by the disjoint union.

We recall the construction of the tensor product of  $M$ -sets. Let  $A$  be a right  $M$ -set and  $B$  a left  $M$ -set. Denote by  $A \otimes_M B$  the set of equivalence classes  $(A \times B) / \sim$ , where  $\sim$  is the equivalence relation generated by  $(am, b) \sim (a, mb)$ . Here  $a \in A, b \in B$  and  $m \in M$ . The class of  $(a, b)$  in  $A \otimes_M B$  is denoted by  $a \otimes b$ .

For any set  $C$ , one has the following bijection

$$\text{Hom}_{\mathcal{S}}(A \otimes_M B, C) \cong \text{Hom}_{\mathcal{S}_M}(A, \text{Hom}_{\mathcal{S}}(B, C)),$$

where  $\mathcal{S}$  simply denotes the category of Sets. It is immediate that  $\text{Hom}_{\mathcal{S}}(B, C)$  has a right  $M$ -set structure, given by  $(f \cdot m)(b) = f(mb)$ ,  $m \in M, b \in B, f \in \text{Hom}_{\mathcal{S}}(B, C)$ . It follows that for any  $M$ -set  $A$ , the functor

$$(-) \otimes_M A : \mathcal{S}_M \rightarrow \mathcal{S}$$

commutes with arbitrary colimits. One can easily check that any covariant functor  $F : \mathcal{S}_M \rightarrow \mathcal{S}$ , commuting with all colimits, is isomorphic to a functor of the type  $(-) \otimes_M A$ . Here,  $A = F(M)$  as a set, while a left  $M$ -set structure on  $A$  is induced by  $ma := F(l_m)(a)$ , where  $a \in F(M), m \in M$  and  $l_m : A \rightarrow A$  is the homomorphism given by  $l_m(a) = ma$ .

**2.1. General remarks on morphisms of topoi.** Recall that a *Grothendieck topos*, or simply a topos, is a category, equivalent to the category of set valued sheaves over a site. A *geometric morphism*, or simply morphism, between topoi  $f : \mathcal{T} \rightarrow \mathcal{T}_1$  consists of a pair of functors  $f^* : \mathcal{T}_1 \rightarrow \mathcal{T}$  and  $f_* : \mathcal{T} \rightarrow \mathcal{T}_1$ , called the *inverse* and *direct image* respectively. These functors must satisfy the following two properties:

- (i)  $f^*$  is the left adjoint of  $f_*$ , (in particular  $f^*$  commutes with all colimits, while  $f_*$  commutes with all limits).
- (ii)  $f^*$  commutes with finite limits.

Clearly, for a given functor  $f^*$  (resp.  $f_*$ ), the adjoint  $f_*$  (resp.  $f^*$ ) is determined uniquely up to an isomorphism.

Topoi and morphisms of topoi form a 2-category. The 2-morphisms (or 2-cells) between morphism are natural transformations between the inverse image functors. This set is in a one-to-one correspondence with the set of natural transformation between inverse image functors.

The category of sets  $\mathcal{S}$  is the terminal object in the 2-category of topoi. A *point* of a topos  $\mathcal{T}$  is a geometric morphism  $\mathcal{S} \rightarrow \mathcal{T}$ . We let  $\mathbf{Pts}(\mathcal{T})$  be the category of points of  $\mathcal{T}$ . Any morphism of topoi  $f = (f^*, f_*) : \mathcal{T} \rightarrow \mathcal{T}_1$  yields the functor  $\mathbf{Pts}(f) : \mathbf{Pts}(\mathcal{T}) \rightarrow \mathbf{Pts}(\mathcal{T}_1)$ .

**2.2. The topos  $\mathcal{S}_M$ .** Recall that for a monoid  $M$ , we denote the category of right  $M$ -sets by  $\mathcal{S}_M$ . Let  $f : M \rightarrow N$  be a monoid homomorphism. If  $A$  is a right  $N$ -set, we can consider it as a right  $M$ -set, denoted  $f^*(A)$ , via the homomorphism  $f$ . Similarly for left  $M$ -sets. In particular,  $N$  itself can be seen as both a right, as well as a left  $M$ -set. Thus, for any right  $M$ -set  $A$ , we can form  $\mathbf{Hom}_{\mathcal{S}_M}(N, A)$  and  $A \otimes_M N$ . Both of these sets have right  $N$ -set structures given by

$$(g \cdot n)(x) = g(nx) \quad \text{and} \quad (a \otimes n)x := a \otimes (nx)$$

respectively, where  $a \in A, n, x \in N$  and  $g \in \mathbf{Hom}_{\mathcal{S}_M}(N, A)$ . These  $N$ -sets are the left and right adjoints of the functor  $f^*$  and we denote them by  $f_!(A)$  and  $f_*(A)$ .

It is a well known fact that the pair  $(f^*, f_*)$  defines a morphism of topoi  $\mathcal{S}_M \rightarrow \mathcal{S}_N$ . The pair  $(f_!, f^*)$  on the other hand need not be a geometric morphism in general. However, if  $N$  is a localisation of  $M$ ,  $(f_!, f^*)$  is a geometric morphism.

**2.3. The points of  $\mathcal{S}_M$  and filtered  $M$ -sets.** A left  $M$ -set  $A$  is called *filtered*, provided the functor

$$(-) \otimes_M A : {}_M\mathcal{S} \rightarrow \mathcal{S}$$

commutes with finite limits. The collection of all filtered left  $M$ -sets is of course a full subcategory of  ${}_M\mathcal{S}$ , which we denote by  ${}_M\mathcal{F}$ . According to Diaconescu's theorem [2, Theorem VII. 2 on p. 381], one has an equivalence of categories

$${}_M\mathcal{F} \xrightarrow{\sim} \mathbf{Pts}(\mathcal{S}_M),$$

which sends a filtered left  $M$ -set  $A$  to the morphism  $f_A : \mathcal{S} \rightarrow \mathcal{S}_M$ . Here

$$(f_A)^* = (-) \otimes_M A : \mathcal{S}_M \rightarrow \mathcal{S} \quad \text{and} \quad (f_A)_* = \mathbf{Hom}_{\mathcal{S}}(A, -) : \mathcal{S} \rightarrow \mathcal{S}_M.$$

The right  $R$ -set structure on  $\mathbf{Hom}_{\mathcal{S}}(A, B)$  is given by  $(f \cdot m)(a) = f(ma)$ . The set of isomorphism classes of left filtered  $M$ -sets will be denoted by  ${}_M\mathbf{F}$ .

The following well-known fact [3, p.24] is a very useful tool for checking whether a given  $M$ -set is filtered.

**Lemma 2.3.1.** *A left  $M$ -set  $A$  is filtered if and only if the following three conditions hold:*

- (F1)  $A \neq \emptyset$ .
- (F2) *If  $m_1, m_2 \in M$  and  $a \in A$  satisfies the condition*

$$m_1 a = m_2 a,$$

*there exist  $m \in M$  and  $\tilde{a} \in A$  such that  $m\tilde{a} = a$  and  $m_1 m = m_2 m$ .*

- (F3) *If  $a_1, a_2 \in A$ , there are  $m_1, m_2 \in M$  and  $a \in A$  such that  $m_1 a = a_1$  and  $m_2 a = a_2$ .*

*Example 2.3.2.* Let  $M = \{1\}$  be the trivial monoid. In this case, our topos is just the category of sets. As such, condition F2 always holds. By F1, any filtered  $M$ -set  $A$  has at least one element, while F3 says that  $A$  has exactly one element. Thus,  $\mathbf{Pts}(\mathcal{S})$  is equivalent to the category with one object and one arrow. So  $\mathcal{S}$  has a unique point, up to a canonical isomorphism.

*Example 2.3.3.* Let  $G$  be a group and  $A$  a filtered  $G$ -set. For any  $a \in A$ , we have the  $G$ -set homomorphism

$$\lambda_a : G \rightarrow A, \quad \lambda_a(g) = ga, \quad a \in A.$$

Assume  $\lambda_a(g) = \lambda_a(h)$ , that is,  $ga = ha$ . By F2 (take  $m_1 = g, m_2 = h$ ) one has  $gm = hm$  for some  $m \in G$ . Since  $G$  is a group, we obtain  $g = h$  and hence  $\lambda_a$  is injective. Take  $x \in A$ . By F3 (take  $a_1 = a, a_2 = x$ ) there are  $m_1, m_2 \in G$  and  $a_0 \in B$ , such that  $m_1 a_0 = a$  and  $m_2 a_0 = x$ . Thus  $x = m_2 m_1^{-1} a = \lambda_b(m_2 m_1^{-1})$  and so  $\lambda_b$  is surjective, showing that the category  $\mathbf{Pts}(\mathcal{S}_G)$  is equivalent to  $G$ , considered as a one object category.

*Example 2.3.4.* Let us consider  $M$  as a left  $M$ -set. The action is obviously given by the multiplication in  $M$ . In fact,  $1 \in M$ , hence F1 holds. To see that F2 is satisfied, take  $m = a$  and  $\tilde{a} = 1$ . Similarly, for F3 we can take  $m_1 = a_1, m_2 = a_2$  and  $a = 1$ . We call this the *trivial point* of  $\mathcal{S}_M$ .

**Lemma 2.3.5.** *Let  $f : M \rightarrow N$  be a homomorphism of monoids. If  $A$  is a filtered  $M$ -set, then*

$$f_!(A) := A \otimes_M N$$

*is a filtered  $N$ -set, where  $N$  is considered as an  $M$ -set via the homomorphism  $f$ . In this way, one obtains a functor*

$$f_! : \mathcal{F}_M \rightarrow \mathcal{F}_N,$$

*and hence a functor*

$$f_* : \mathbf{Pts}(\mathcal{S}_M) \rightarrow \mathbf{Pts}(\mathcal{S}_N).$$

*Proof.* Take  $X$  to be an  $N$ -set. We need to show that the functor

$$X \mapsto f_!(A) \otimes_N X$$

commutes with finite limits. But this functor is the same as

$$X \mapsto (A \otimes_M N) \otimes_N X = A \otimes_M f^*(X).$$

Since both functors  $f^* : \mathcal{F}_N \rightarrow \mathcal{F}_M$  and  $A \otimes_M (-) : \mathcal{F}_M \rightarrow \mathcal{F}_M$  commute with finite limits, the composite functor also commutes.  $\square$

### 3. THE CASE OF COMMUTATIVE MONOIDS

From here on onwards, we assume that  $M$  is a commutative monoid. As such, we will no longer make any distinction between left and right  $M$ -sets. The category of  $M$ -sets will be written as  $\mathcal{S}_M$  and the action of  $M$  on  $A$  will be written as  $ma$ .

**Proposition 3.0.6.** *Let  $M$  be a monoid and  $A, B$  filtered  $M$ -sets. Then  $A \otimes_M B$  is a filtered  $M$ -set as well.*

*Proof.* We have the commutative diagram

$$\begin{array}{ccc} \mathcal{S}_M & \xrightarrow{B \otimes_M (-)} & \mathcal{S}_M \\ & \searrow (A \otimes_M B) \otimes (-) & \downarrow A \otimes_M (-) \\ & & \mathcal{S} \end{array}$$

Since the functor  $B \otimes_M (-)$ ,  $A \otimes_M (-)$  commutes with finite limits and colimits, so does the composition  $(A \otimes_M B) \otimes_M (-)$ .  $\square$

### 3.1. From prime ideals to topos points.

**Lemma 3.1.1.** *Let  $\mathfrak{p}$  be a prime ideal of  $M$ . The localization  $M_{\mathfrak{p}}$  is a filtered  $M$ -set, where  $M_{\mathfrak{p}}$  is considered as an  $M$ -set via the canonical monoid homomorphism  $M \rightarrow M_{\mathfrak{p}}$ .*

*Proof.* Since  $1 \in M$ , the set  $M_{\mathfrak{p}}$  is not empty. Hence, F1 holds for  $M_{\mathfrak{p}}$ . To show that F2 holds, take  $m_1, m_2 \in M$  and  $b \in M_{\mathfrak{p}}$  such that  $m_1 b = m_2 b$ . Thus, we can write  $b = \frac{m_3}{s}$ , where  $m_3, s \in M$  and  $s \notin \mathfrak{p}$ . Since  $\frac{m_1 m_3}{s} = \frac{m_2 m_3}{s}$ , there is an element  $t \notin \mathfrak{p}$  such that  $m_1 m_3 s t = m_2 m_3 s t$ . So, we can take  $m = m_3 s t$  and  $\tilde{b} = \frac{1}{s^2 t}$ . To show F3, take  $b_1, b_2 \in M_{\mathfrak{p}}$ . We can write  $b_1 = \frac{m'_1}{s_1}$  and  $b_2 = \frac{m'_2}{s_2}$ . Put  $b = \frac{1}{s_1 s_2}$ ,  $m_1 = m'_1 s_2$  and  $m_2 = m'_2 s_1$ . Then  $m_1 b = b_1$  and  $m_2 b = b_2$ , proving the assertion.  $\square$

This lemma shows that we have a map  $\gamma_M : \text{Spec}(M) \rightarrow F_M$ , where  $\text{Spec}(M)$  denotes the prime ideals of a commutative monoid  $M$  (see for example [1]). However, as previously shown [4],  $\text{Spec}(M)$  is in fact a lattice. As such, it has two operations which make it into a commutative monoid. One is simply the union (note that in the monoid world, the union of ideals is again an ideal) and the other is done as follows: For two prime ideals  $\mathfrak{p}$  and  $\mathfrak{q}$ , define  $\mathfrak{p} \frown \mathfrak{q}$  to be the union of all the prime ideals contained in both  $\mathfrak{p}$  and  $\mathfrak{q}$ . That is,

$$(3.1.1) \quad \mathfrak{p} \frown \mathfrak{q} := \bigcup_i \mathfrak{p}_i \mid \mathfrak{p}_i \subset \mathfrak{p} \cap \mathfrak{q}.$$

We showed in Lemma 3.0.6 and Example 2.3.4 that  $F_M$  was in fact a commutative monoid. It is straightforward to see that  $M_{\mathfrak{p}} \otimes_M M_{\mathfrak{q}} \cong M_{\mathfrak{p} \frown \mathfrak{q}}$ . Hence  $\gamma_M$  is a monoid homomorphism. Clearly,  $M_{\mathfrak{p}} \cong M_{\mathfrak{q}}$  as monoids if and only if they are isomorphic as  $M$ -sets. We sum up our discussion in the following proposition.

**Proposition 3.1.2.** *Let  $M$  be a commutative monoid. There exists an injective homomorphism of commutative monoids*

$$\gamma_M : \text{Spec}(M) \rightarrow F_M,$$

*given by  $p \mapsto M_p$ .*

**3.2. From topos points to prime ideals.** The aim of this subsection is to prove that the morphism  $\gamma_M$ , defined above, is an isomorphism if  $M/M^*$  is finitely generated. Here  $M^*$  denotes the subgroup of invertible elements.

**Lemma 3.2.1.** *Let  $A$  be an  $M$ -set. Define the set  $\mathfrak{p}_A$ , or simply  $\mathfrak{p}$  when there is no ambiguity, by*

$$\mathfrak{p} = \{m \in M \mid l_m : A \rightarrow A \text{ is not an isomorphism}\}.$$

*Then  $\mathfrak{p}$  is a prime ideal of  $M$ . Moreover,  $A$  is an  $M_{\mathfrak{p}}$ -set.*

*Proof.* First we show that  $\mathfrak{p}$  is an ideal. Take  $m \in \mathfrak{p}$ ,  $x \in M$  and assume  $l_{xm}$  is an isomorphism. Then  $l_{xm} \circ g = id = g \circ l_{xm}$ , for some  $g : A \rightarrow A$ , which is an  $M$ -set isomorphism. The last condition implies that  $l_x \circ g = g \circ l_x$ . Since  $l_{xm} = l_x \circ l_m$ , one obtains  $(g \circ l_x) \circ l_m = id$ . On the other hand,  $l_m \circ (g \circ l_x) = l_m \circ l_x \circ g = l_{xm} \circ g = id$ . Thus  $l_m$  is an isomorphism. This contradicts our assumption. Hence,  $l_{xm}$  is not an isomorphism and so  $xm \in \mathfrak{p}$ .

To show that  $\mathfrak{p}$  is prime, take  $m, n \notin \mathfrak{p}$ . Since,  $l_m$  and  $l_n$  are isomorphisms, so is their composition  $l_{mn}$ , and thus  $mn \notin \mathfrak{p}$ .

For the last statement take an element  $a \in A$  and  $x = \frac{m}{s} \in M_{\mathfrak{p}}$ . Since  $s \notin \mathfrak{p}$ , the map  $l_s$  is an isomorphism. We put

$$xa := ml_s^{-1}(a).$$

In this way one obtains an action of  $M_{\mathfrak{p}}$  on  $A$ .  $\square$

**3.2.1. Conservative  $M$ -sets.** We call an  $M$ -set  $A$  *conservative*, provided any  $m \in M$ , such that  $l_m : A \rightarrow A$  is an isomorphism, is an invertible element of  $M$ .

**Lemma 3.2.2.** *Let  $A$  be an  $M$ -set. Then  $A$  is conservative as an  $M_{\mathfrak{p}}$ -set, where  $\mathfrak{p} = \mathfrak{p}_A$  is the same as in Lemma 3.2.1.*

*Proof.* Any element  $x \in M_{\mathfrak{p}}$  can be written as  $x = \frac{m}{s}$ , where  $s \notin \mathfrak{p}$ . We have  $l_x = l_s^{-1} \circ l_m$ . Assume  $l_x : A \rightarrow A$  is an isomorphism. Then  $l_m = l_s \circ l_x$  is also an isomorphism. Thus,  $m \notin \mathfrak{p}$  and therefore  $x$  is invertible in  $M_{\mathfrak{p}}$ .  $\square$

**3.2.2. Source.** Let  $A$  be an  $M$ -set. We call an element  $a \in A$  a *source*, provided  $a = mb$ ,  $m \in M$ ,  $b \in A$  implies that  $l_m : A \rightarrow A$ , given by  $b \mapsto mb$ ,  $b \in A$ , is an isomorphism.

**Lemma 3.2.3.** *Let  $\mathfrak{p}$  be a prime ideal of  $M$  and assume that  $M_{\mathfrak{p}}$  is a conservative  $M$ -set. Then  $M_{\mathfrak{p}} \cong M$  and 1 is a source of  $M_{\mathfrak{p}}$ .*

*Proof.* Take  $s \in M \setminus \mathfrak{p}$ . Since  $s$  is invertible in  $M_{\mathfrak{p}}$ , the homomorphism  $l_s : M_{\mathfrak{p}} \rightarrow M_{\mathfrak{p}}$ ,  $x \mapsto sx$  is an isomorphism. The definition of conservativeness implies that  $s$  is invertible in  $M$  and as such  $M \cong M_{\mathfrak{p}}$ . It is no clear that the unit, indeed every invertible element, is a source.  $\square$

**Lemma 3.2.4.** *Let  $A$  be a filtered and conservative  $M$ -set. If  $A$  possesses a source  $a \in A$ , the map*

$$\lambda_a : M \rightarrow A, \quad \lambda_a(m) = ma$$

*is an isomorphism.*

*Proof.* Take  $b \in A$ . By property F3, there are  $c \in A$ ,  $m_1, m_2 \in A$  such that  $a = m_1c$  and  $b = m_2c$ . Since  $a$  is a source and  $A$  is a conservative  $M$ -set, it follows that  $m_1$  is invertible. Hence  $b = m_2m_1^{-1}a$  and so  $\lambda_a$  is surjective. If  $\lambda_a(m_1) = \lambda_a(m_2)$ , then  $m_1a = m_2a$ . By F2, there are elements  $b \in A$  and  $m \in M$  such that  $a = mb$  and  $m_1m = m_2m$ . Since  $a$  is a source and  $A$  conservative,  $m$  must be invertible and thus  $m_1 = m_2$ . So  $\lambda_a$  is injective and hence an isomorphism.  $\square$

**Lemma 3.2.5.** *Let  $A$  be a filtered  $M$ -set and  $m \in M$ . If the map  $l_m : A \rightarrow A$ ,  $a \mapsto ma$  is surjective, it is an isomorphism.*

*Proof.* If  $a_1, a_2 \in A$  and  $ma_1 = ma_2$  holds, then by F3 there exist  $a \in A$  and  $m_1, m_2 \in M$  such that  $m_1a = a_1$ ,  $m_2a = a_2$ . Thus,  $mm_1a = mm_2a$ . F2 implies that there exist an element  $b \in A$  and  $m_3 \in M$  such that  $a = m_3b$  and  $mm_1m_3 = mm_2m_3$ . Since  $l_m$  is surjective, there exist  $c \in A$  such that  $b = mc$ . We have

$$a_1 = m_1a = m_1m_3b = m_1m_3mc = m_2m_3mc = m_2m_3b = m_2a = a_2.$$

Hence  $l_m$  is injective and as such an isomorphism.  $\square$

We will say that  $M$  is *almost finitely generated* if there are a finite number of non-invertible elements  $c_1, \dots, c_k \in M \setminus M^*$ , such that any  $m \in M$  can be written as a product  $m = ge_1^{\alpha_1} \dots e_k^{\alpha_k}$ . Here  $g \in M^*$  is invertible and  $\alpha_1, \dots, \alpha_k \in \mathbb{N}$  are natural numbers. The elements  $e_1, \dots, e_k$  are called *almost generators*. Clearly, this class encompasses any group as well as any finitely generated monoid. Moreover, if  $M$  is almost finitely generated and  $\mathfrak{p}$  is a prime ideal of  $M$ , then  $M_{\mathfrak{p}}$  is also almost finitely generated.

An other way of saying that  $M$  is almost finitely generated is that  $M/M^*$  is finitely generated.

**Lemma 3.2.6.** *Let  $M$  be an almost finitely generated commutative monoid and  $A$  a filtered  $M$ -set which is also conservative. There exists an element  $a \in A$  such that the canonical map  $\lambda_a : M \rightarrow A$ ,  $m \mapsto ma$  is an isomorphism of  $M$ -sets.*

*Proof.* Let  $e_1, \dots, e_k$  be almost generators of  $M$ . Since  $A$  is conservative, the map  $l_{e_i} : A \rightarrow A$  is not an isomorphism for any  $i = 1, \dots, k$ . It follows from Lemma 3.2.5 that the maps  $l_{e_i}$  are not surjective. Hence, for every  $i$ , there is an element  $a_i \in A$ , such that  $a_i$  is not in the image of  $l_{e_i}$ . As such, the equations  $e_i x = a_i$  have no solutions in  $A$ . From F3 we know that for the collection  $a_1, \dots, a_k$ , there exists  $a \in A$  and  $m_1, \dots, m_k \in M$  such that

$$a_i = m_i a, \quad i = 1, \dots, k.$$

We claim that the element  $a$  is a source. In fact, if  $a = mb$ , where  $b \in A$  and  $m \in M$ , our assumption that  $M$  is almost finitely generated implies that the element  $m$  can be written in the form

$$m = ge_1^{\alpha_1} \dots e_k^{\alpha_k}.$$

We claim that  $\alpha_1 = \dots = \alpha_k = 0$ . If not, without loss of generality,  $\alpha_1 > 0$ . We have  $m = e_1 m'$ . Thus,  $a_1 = m_1 a = m_1 mb = m_1 (e_1 m') b = e_1 x$ , where  $x = m_1 m' b$ . This contradicts the assumption that  $a = e_1 x$  has no solution. We have shown that  $m = g$  is invertible in  $M$ , and so it follows that  $a$  is a source. Since  $M$  is conservative,  $\lambda_a : M \rightarrow A$  is an isomorphism by Lemma 3.2.4.  $\square$

We are now in position to prove our main theorem.

**Theorem 3.2.7.** *Let  $M$  be an almost finitely generated commutative monoid. There exists an isomorphism of commutative monoids*

$$\gamma_M : \text{Spec}(M) \rightarrow F_M,$$

*given by  $p \mapsto M_p$ . In particular, if  $f : \mathcal{S} \rightarrow \mathcal{S}_M$  is a point of the topos  $\mathcal{S}_M$ , there exist a prime ideal  $\mathfrak{p}$ , such that the functors*

$$f^* \quad \text{and} \quad (-)_{\mathfrak{p}} : \mathcal{S}_M \rightarrow \mathcal{S}$$

*are isomorphic.*

*Proof.* Let  $A$  be a filtered  $M$ -set. Take  $\mathfrak{p} = \mathfrak{p}_A$  to be the prime ideal defined in Lemma 3.1.1. Then  $A$  becomes an  $M_{\mathfrak{p}}$ -set, which is conservative and filtered over  $M_{\mathfrak{p}}$ . Hence, Lemma 3.2.6 implies that  $A$  is isomorphic to  $M_{\mathfrak{p}}$  and we are done.  $\square$

#### 4. INFINITELY GENERATED COMMUTATIVE MONOIDS

For a commutative monoid  $M$ , we can consider  $M^{sl} = M / \sim$ , where  $\sim$  is the congruence generated by  $m^2 = m$  for every  $m \in M$ . This is the universal semilattice associated to  $M$ . We have the following result:

**Proposition 4.0.8.** *Let  $M$  be a commutative monoid. There are injective homomorphisms of commutative monoids*

$$M^{sl} \xrightarrow{\alpha_M} \text{Spec}(M) \xrightarrow{\gamma_M} F_M,$$

*which are isomorphisms when  $M$  is finitely generated. The monoidal structures are induced by the quotient, the operation  $\boxtimes$  and the tensor product respectively.*

We note that  $\text{Spec}$  is a contravariant functor, where as both  $sl$  and  $F$  are covariant functors. However, when  $M$  is not finitely generated, neither  $\alpha_M$  nor  $\gamma_M$  are isomorphisms, as we shall see from the following example.

*Example 4.0.9.* Let  $M = \mathbb{N}^{\text{mult}} = \bigoplus_{i=1}^{\infty} \mathbb{N}$  be the natural numbers under multiplication. For simplicity, we will use the additive notation in this example.

i) To see that  $\alpha_M$  is not an isomorphism, we first observe that since  $M$  has countable elements, so does  $M^{sl}$ . As previously shown [4, Lemma 2.1], we have an isomorphism

$$\text{Spec}(M) \cong \text{Hom}(M, \mathbb{I}).$$

Here  $\mathbb{I} = \{0, 1\}$  with obvious multiplication. Since  $M$  is free with countable generators,  $\text{Spec}(M)$  is uncountable.

ii) To see that  $\gamma_M$  is not an isomorphism, consider

$$A = \{(a_0, a_1, \dots, a_n, \dots) \mid a_n \in \mathbb{Z}, a_n \geq -n\} \subset \bigoplus_{i=1}^{\infty} \mathbb{Z}.$$

The  $M$ -set structure of  $A$  is given by componentwise addition. We claim that this is a filtered  $M$ -set. It is clear that this is non-empty. For every element  $a$ , we have

$$a = (a_0, a_1, \dots, a_n, 0, \dots) = (a_0, a_1 + 1, \dots, a_n + n, 0, \dots) + (0, -1, \dots, -n, \dots).$$

As such, condition F2 holds. For F3, observe that  $m_1 + a = m_2 + a$  implies  $m_1 = m_2$ ,  $m_1, m_2 \in M, a \in A$ .

Let  $m \in M$  and assume  $l_m : A \rightarrow A$ ,  $l_m(a) = am$  is an isomorphism. In particular,  $l_m$  is surjective. So  $(0, -1, -2, \dots)$  is in the image of  $l_m$ . As such, there exists  $x = (x_0, x_1, x_2, \dots) \in A$  such that

$$(m_0, m_1, m_2, \dots) + (x_0, x_1, x_2, \dots) = (0, -1, -2, \dots).$$

This implies that  $m = (m_0, m_1, m_2, \dots) = (0, 0, 0, \dots) = 0 \in M^*$  is invertible in  $M$  and as such  $A$  is conservative.

For any  $a \in A$ , we have

$$(a_0, a_1, \dots, a_n, 0, 0, \dots) = (0, 0, \dots, 0, 1, 0, \dots) + (a_0, a_1, \dots, a_n, -1, 0, \dots).$$

We see that  $m = (0, 0, \dots, 0, 1, 0, \dots)$  is not invertible in  $M$  and so  $A$  has no source. Lemma 3.2.2 now shows that  $A$  is not a localisation of  $M$  with a prime ideal. Indeed, [1, Lemma 1.1] shows that  $A$  is not a localisation of  $M$  of any kind.

## REFERENCES

- [1] G. Cortinas, C. Haesemeyer, M.E. Walker and C. Weibel. Toric varieties, monoid schemes and *cdh* descent. *Journal für die reine und angewandte Mathematik (Crelles Journal)* 2015.698 (2015): 1-54.
- [2] S. Mac Lane and I. Moerdijk. *Sheaves in geometry and logic. A first introduction to topos theory.* Corrected reprint of the 1992 edition. Universitext. Springer-Verlag, New York, 1994. xii+629 pp.
- [3] I. Moerdijk. *Classifying Spaces and Classifying Topoi.* Lecture Notes in Mathematics. Springer-Verlag, Berlin. Vol.1616. 1995.
- [4] I. Pirashvili. On the spectrum of monoids and semilattices. *Journal of Pure and Applied Algebra* 217 (2013) 901-906.

*E-mail address:* `ilia_p@ymail.com`